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Non-holonomic geodesic flows on Lie groups and the integrable Suslov problem on *SO*(4)

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Abstract. We study geodesic flows on Lie groups with the left-invariant non-holonomic constraint. In the case of the existence of an invariant measure, we find new integrable non-Hamiltonian systems on $SO(4)$ and other six-dimensional Lie groups.

0. Introduction

Hertz has classified mechanical systems with n degrees of freedom and $k < n$ linear constraints into the holonomic and non-holonomic according to whether constraints are integrable or not. He noticed that non-holonomic equations (derived from the d'Alambert–Lagrange principle) are not Hamiltonian [1]. It is well known that a Hamiltonian system is integrable if it has n integrals in involution. By the classical Liouville theorem, under the compactness assumption, the motion in the $2n$-dimensional phase space could be seen as the winding on n-dimensional invariant tori [2]

$$\varphi_i = \omega_i t + \varphi_{0i} \operatorname{mod}(2\pi) \qquad \omega_i = \text{constant},\ i = 1, \ldots, n.$$

In general, we need $2n-k-1$ integrals of motion for integrating the non-holonomic system. In some solvable problems the behaviour of the system is close to the Hamiltonian integrable system, we need 'only' $2n-k-2$ integrals, and trajectories in the phase space belong to invariant two-dimensional tori. This is a consequent of the existence of an invariant measure: by using an integrating factor it is possible to find locally one more integral of motion [3, 4]. Moreover, if the invariant manifold is compact, connected, and equations have no singularity upon it, then the invariant manifold is diffeomorphic to a 2-torus. By Kolmogorov's theorem [5] on the reduction of differential equations with smooth invariant measure on the torus, there exist angular coordinates $\varphi_1, \varphi_2 \operatorname{mod}(2\pi)$ in which motion takes the form

$$\dot{\varphi}_1 = \frac{\omega_1}{\Phi(\varphi_1, \varphi_2)} \qquad \dot{\varphi}_2 = \frac{\omega_2}{\Phi(\varphi_1, \varphi_2)}$$

where ω_1, ω_2 are constants and Φ is a smooth positive function. The reduction of non-holonomic systems with symmetry, as well methods of integration of systems with an invariant measure, can be found in [1, 3, 4, 6, 7].

Veselov and Veselova considered non-holonomic geodesic flows on Lie groups with left-invariant metrics [6]. They specified a right-invariant constraint. Similar generalizations have been applied to systems with left-invariant constraints (Euler–Poincaré–Suslov

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equations) by Kozlov [8] (also see [9]). Following [6, 8], in this paper, we shall make a careful study of the Euler–Poincaré–Suslov equations for six-dimensional Lie groups.

In section 1 we shall set the notation and definitions. An example of the construction of angular coordinates for the basic integrable case is given in section 2. In section 3 we shall prove the main result of the paper: the integrability of non-holonomic geodesic flows with an invariant measure on two classes of Lie groups.

1. Euler–Poincaré equations with constraint

Let Q be the n-dimensional manifold, $L(\dot{x}, x)$ a Lagrangian function and let $D \subset TQ$ be the non-integrable distribution of a tangent bundle. The smooth path $x(t), t \in \Delta$, is admissible (allowed by the constraint) if $\dot{x} \in D_x, t \in \Delta$. The admissible path is a motion of the non-holonomic Lagrangian system (Q, L, D) if it satisfies the d'Alambert–Lagrange principle [1]

$$\left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}, \xi\right) = 0 \qquad \text{for all } \xi \in D_x.$$

Now, let Q be a real Lie group G, $\mathcal{G} = T_eG$ its Lie algebra, and $\mathcal{G}^* = T_e^*G$ the dual vector space of $\mathcal{G}$. Let $\langle .\,, .\rangle$ be the left-invariant metric on G given with the symmetric operator $I : \mathcal{G} \to \mathcal{G}^*$ and let $A = I^{-1} : \mathcal{G}^* \to \mathcal{G}$. If $g(t)$, $t \in \Delta$ is a smooth path, as usual [2], we introduce $\omega(t) = (L_{g-1})_*\dot{g} \in \mathcal{G}$, $M(t) = I\omega(t) \in \mathcal{G}^*$. Then the metric is $\langle \dot{g}, \dot{g}\rangle = (I\omega, \omega) = (M, AM)$.

We shall consider the non-holonomic geodesic flow on G with the constraint defined by the left-invariant 1-form $\alpha(N = (L_g)^*\alpha = \text{constant})$:

$$(\alpha, \dot{g}) = (\alpha, (L_g)_*\omega) = ((L_g)^*\alpha, \omega) = (N, \omega) = (N, AM) = 0 \tag{1}$$

i.e. inertial motion of a mechanical system with the configuration space G, kinetic energy $\frac{1}{2}\langle \dot{g}, \dot{g}\rangle$ and the constraint (1). Equations of the motion, derived from the d'Alambert–Lagrange principle, are reduced to $\mathcal{G}^*$ (or precisely, they are reduced to $(N, AM) = 0 \subset \mathcal{G}^*$):

$$\dot{M} = \mathrm{ad}^*_{dH}M + \lambda N \qquad (N, \omega) = (N, AM) = 0 \tag{2}$$

where the Hamiltonian if $H = \frac{1}{2}(M, AM)$ and ad^*_ξ: $\mathcal{G}^* \to \mathcal{G}^*$, $\xi \in \mathcal{G}$ is the co-adjoint action of the Lie algebra $\mathcal{G}$ on $\mathcal{G}^*$: $(\mathrm{ad}^*_\xi M, \eta) = (M, [\xi, \eta])$ for all $\eta \in \mathcal{G}$, $M \in \mathcal{G}^*$ [2, 7]. From the constraint we can find the Lagrange multiplier

$$\lambda = -(N, A(\mathrm{ad}^*_{AM}M))/(N, AN) = (M, [AN, AM])/(N, AN). \tag{3}$$

Let $e_1, \ldots, e_n$ be the base of the Lie algebra $\mathcal{G}$ with structural constants $[e_i, e_j] = \sum_k C^k_{ij}e_k$, and let $e^1, \ldots, e^n$ be the dual base of $\mathcal{G}^*$. Also, let ω^j, M_i be coordinates of ω and M according to those bases, and let $A^{ij} = (e^i, Ae_j)$. With such a notation, coordinately equations (2) take the form

$$\dot{M}_k = \{M_k, H\} + \lambda N_k = \sum_{i,j,l} C^l_{ik}M_lA^{ij}M_j + \lambda N_k \qquad \sum_{i,j} N_iA^{ij}M_j = 0 \tag{4}$$

where Lie–Poisson brackets on $\mathcal{G}^*$ are

$$\{F, G\} = \sum_{i,j,l} -C^l_{ij}M_l\partial_iF\partial_jG \qquad F, G \in C^\infty(\mathcal{G}^*).$$

For the case $G = SO(3)$, (4) becomes

$$\dot{M} = M \times AM + \lambda N \qquad (N, \omega) = (N, AM) = 0 \tag{5}$$

where × is a usual vector product in R^3. Equations (5) describe rotation of a rigid body fixed at a point and subject to the non-integrable constraint $(N, \omega) = 0$. N is the constant vector, ω is the angular velocity, M the angular momentum in body coordinates and $I = A^{-1}$ is the inertia operator of a rigid body. This problem was first studied and solved by Suslov [10]. Thus, equations (2) are called *Euler–Poincaré–Suslov* (EPS) equations.

Generalizations of the Suslov problem, supposing that the body rotates in an axially-symmetric potential force field, can be found in [3, 4, 11].

2. Conditions for the existence of an integral invariant—example

Kozlov gave necessary and sufficient conditions for the existence of an invariant measure of EPS equations in the case of compact groups [8]. We shall need similar results for non-compact groups as well. It can be proved that equations (2) have an invariant measure if and only if

$$K\,\mathrm{ad}^*_{AN} N + T = \mu N \qquad \mu \in R \tag{6}$$

where $K = 1/(N, AN)$, $T \in \mathcal{G}^*$, $(T, \xi) = \mathrm{Tr}(\mathrm{ad}_\xi)$, $\xi \in \mathcal{G}$, or in coordinate notation

$$K \sum_{i,g,k} C^k_{ij} A^{ig} N_g N_k + \sum_k C^k_{jk} = \mu N_j \qquad \mu \in R.$$

If we have $n - 3$ integrals, then the EPS equations, with an integral invariant, are integrable. The Hamiltonian function is always the first integral. Thus, we need two additional integrals (independent of the constraint and Hamiltonian) for six-dimensional groups.

The following lemma is a modification of the well known involutive condition on a function to be the integral in Hamiltonian systems.

Lemma 1. If F satisfies $\{F, H\} + \lambda\, \mathrm{d}F(N)|_{(N,AM)=0} = 0$ (λ is given with (3)) then F is the integral of equations (2). In particular, all invariant I on $\mathcal{G}^*$ with the condition $\mathrm{d}I(N)|_{(N,AM)=0} = 0$ are integrals of (2).

To illustrate the behaviour of integrable systems we start with the following example on $SO(4)$.

Example 1. We can choose the base $e^{\pm}_i$, $i = 1, 2, 3$, in which $\omega \in SO(4)$ has the following representation

$$\omega = \sum_{i=1}^{3} (\omega^i e^+_i + \omega^{i+3} e^-_i) = \begin{pmatrix} 0 & -\omega^3 & \omega^2 & -\omega^4 \\ \omega^3 & 0 & -\omega^1 & -\omega^5 \\ -\omega^2 & \omega^1 & 0 & -\omega^6 \\ \omega^4 & \omega^5 & \omega^6 & 0 \end{pmatrix}.$$

Then $[e^+_i, e^+_j] = \varepsilon_{ijk} e^+_k$, $[e^-_i, e^-_j] = \varepsilon_{ijk} e^+_k$, $[e^+_i, e^-_j] = \varepsilon_{ijk} e^-_k$, and invariants on $so(4)^*$ become $I_1 = \sum_{k=1}^{6} M_k^2$, $I_2 = \sum_{k=1}^{3} M_k M_{k+3}$. For diagonal metrics, the Hamiltonian is $H = \frac{1}{2} \sum_{i=1}^{6} A^i M_i^2$ and if we define $A_{ij} = A^i - A^j$, equations (4) take the form

$$\begin{aligned}
\dot{M}_1 &= M_2 M_3 A_{32} + M_5 M_6 A_{65} + \lambda N_1 & \dot{M}_4 &= M_5 M_3 A_{35} + M_2 M_6 A_{62} + \lambda N_4 \\
\dot{M}_2 &= M_3 M_1 A_{13} + M_6 M_4 A_{46} + \lambda N_2 & \dot{M}_5 &= M_3 M_4 A_{43} + M_6 M_1 A_{16} + \lambda N_5 \\
\dot{M}_3 &= M_1 M_2 A_{21} + M_4 M_5 A_{54} + \lambda N_3 & \dot{M}_6 &= M_4 M_2 A_{24} + M_1 M_5 A_{51} + \lambda N_6
\end{aligned} \tag{7}$$

$$\sum_{k=1}^{6} A^k N_k M_k = 0.$$

The basic integrable non-holonomic example is N being the eigenvector of the operator A. Without loss of generality, suppose $N = e^3_-$. Then equations (7) preserve the measure on $(N, \omega) = \omega^6 = A^6 M_6 = 0 \subset so(4)^*$ and they have three independent first integrals: the invariant I_1, and two new ones

$$F_2 = A_{13}M_1^2 - A_{32}M_2^2 \qquad F_3 = A_{43}M_4^2 - A_{35}M_5^2. \tag{8}$$

The Hamiltonian is a linear combination of I_1, F_2 and F_3. We define new variables u and v by

$$u = A_{13}M_1^2 + A_{32}M_2^2 \qquad v = A_{43}M_4^2 + A_{35}M_5^2. \tag{9}$$

For the sake of simplicity, we suppose $A_{13}, A_{23}, A_{43}, A_{53} > 0$. On the invariant manifold

$$M_c = \{M \in so(4)^* | I_1 = c_1, F_2 = c_2, F_3 = c_3\}$$

equations (7) do not have a singularity (for example, if c_1 is big enough). Thus, M_c is diffeomorphic to the 2-torus. In variables u^*, v equations (7) on M_c are

$$\begin{aligned} \dot{u} &= \pm\sqrt{A_{13}A_{23}}\sqrt{c_2^2 - u^2}M_3(u, v) \qquad && -c_2 \leqslant u \leqslant c_2 \\ \dot{v} &= \pm\sqrt{A_{43}A_{53}}\sqrt{c_3^2 - v^2}M_3(u, v) \qquad && -c_3 \leqslant v \leqslant c_3 \\ M_3^2 &= c_1 - \left(\frac{u + c_2}{2A_{13}} + \frac{u - c_2}{2A_{32}} + \frac{v + c_3}{2A_{43}} + \frac{v - c_3}{2A_{35}}\right) \neq 0. \end{aligned} \tag{10}$$

We shall introduce angular variables φ_1, $\varphi_2 \bmod(2\pi)$ with formulae

$$\varphi_1 = \int_{-c_2}^{u} \frac{\mathrm{d}z}{\pm\sqrt{c_2^2 - z^2}} \qquad \varphi_2 = \int_{-c_3}^{v} \frac{\mathrm{d}z}{\pm\sqrt{c_3^2 - z^2}}. \tag{11}$$

The sign (positive or negative) in the integrals depends on whether $u(v)$ increases or decreases. In angular coordinates φ_1 φ_2 the motion on the torus M_c gets the form

$$\dot{\varphi}_1 = \frac{\omega_1}{\Phi(\varphi_1, \varphi_2)} \qquad \dot{\varphi}_2 = \frac{\omega_2}{\Phi(\varphi_1, \varphi_2)} \tag{12}$$

where $\omega_1 = 2\sqrt{A_{13}A_{23}}$, $\omega_2 = 2\sqrt{A_{43}A_{53}}$, $\Phi(\varphi_1, \varphi_2) = M_3^{-1}(\varphi_1, \varphi_2)$, according to Kolmogorov's theorem. It is interesting that if trajectories are closed on one torus M_c, then they are closed on all tori, and this happens when $\sqrt{A_{13}A_{23}/A_{43}A_{53}}$ is a rational number.

3. Integrable non-Hamiltonian systems on *SO*(4) and other six-dimensional Lie groups

Now, we adapt this approach to derive the integrability for a more general situation. We are going to consider two classes $\mathcal{A}$ and $\mathcal{B}$ of six-dimensional Lie algebras $\mathcal{G}$, in which there are bases $e_i^{\pm}$ and $f_i^{\pm}$, $i = 1, 2, 3$, with commutators [12]

class $\mathcal{A}$

$$[e_i^+, d_j^+] = n_k\varepsilon_{ijk}e_k^+ \qquad [e_i^-, e_j^-] = qn_k\varepsilon_{ijk}e_k^+ \qquad [e_i^+, e_j^-] = n_k\varepsilon_{ijk}e_k^- \tag{13}$$

class $\mathcal{B}$

$$[f_i^+, f_j^+] = n_k\varepsilon_{ijk}f_k^+ \qquad [f_i^-, f_j^-] = m_k\varepsilon_{ijk}f_k^- \qquad [f_i^+, f_j^-] = 0 \tag{14}$$

where n_k, m_k and q are (structural) constants.

In class $\mathcal{A}$ there are Lie algebras $so(4)$ ($n_k = 1, q = 1$), $so(3.1)$ ($n_1 = n_2 = 1$, $n_3 = -1, q = -1$), $so(2.2)$ ($n_1 = n_2 = 1$, $n_3 = -1, q = 1$), $e(3)$ ($n_k = 1, q = 0$), $l(3)$ ($n_1 = n_2 = 1, n_3 = -1, q = 0$) etc. $E(3)$ and $L(3)$ are groups of motions of the three-dimensional Euclidean and the pseudo-Euclidean spaces. In class $\mathcal{B}$ there are Lie algebras $so(4) = so(3) \oplus so(3)$ ($n_k = 1, m_k = 1$), $sl(2,R) \oplus sl(2,R)$ ($n_1 = n_2 = m_1 = m_2 = 1, n_3 = m_3 = -1$) etc. Bases $e_i^\pm$ and $f_i^\pm$, for the group $SO(4)$ are related by $f_i^\pm = \frac{1}{2}(e_i^+ \pm e_i^-)$.

Let $e_\pm^i$ and f_+^i be dual bases in $\mathcal{G}^*$, and let $M_i^\pm$, $i = 1, 2, 3$, be coordinates of $M \in \mathcal{G}^*$ according to those bases. Then invariants on $\mathcal{G}^*$ are

class $\mathcal{A}$

$$I_1 = \sum_{k=1}^{3}(qn_k(M_k^+)^2 + n_k(M_k^-)^2) \qquad I_2 = \sum_{k=1}^{3} n_k M_k^+ M_k^- \tag{15}$$

class $\mathcal{B}$

$$I_1 = \sum_{k=1}^{3} n_k(M_k^+)^2 \qquad I_2 = \sum_{k=1}^{3} m_k(M_k^-)^2. \tag{16}$$

The general metric is given with the Hamiltonian

$$H = \tfrac{1}{2}(B_+M^+, M^+) + (CM^+, M^-) + \tfrac{1}{2}(B_-M^-, M^-) \tag{17}$$

where B_+ and B_- are symmetric and C arbitrary. For

$$\omega_+ = \frac{\partial H}{\partial M^+} = B_+M^+ + CM^- \qquad \omega_- = \frac{\partial H}{\partial M^-} = B_-M^- + CM^+ \tag{18}$$

we can write equations (4) as follows [12],

class $\mathcal{A}$

$$\begin{aligned}
&\dot{M}^+ = \bar{M}^+ \times \omega_+ + \bar{M}^- \times \omega_- + \lambda N^+\\
&\dot{M}^- = \bar{M}^- \times \omega_+ + q\bar{M}^+ \times \omega_- + \lambda N^-\\
&\bar{M}_k^+ = n_k M_k^+ \qquad \bar{M}_k^- = n_k M_k^- \qquad k = 1, 2, 3
\end{aligned} \tag{19}$$

class $\mathcal{B}$

$$\begin{aligned}
&\dot{M}^+ = \bar{M}^+ \times \omega_+ + \lambda N^+\\
&\dot{M}^- = \bar{M}^- \times \omega_- + \lambda N^-\\
&\bar{M}_k^+ = n_k M_k^+ \qquad \bar{M}_k^- = m_k M_k^- \qquad k = 1, 2, 3
\end{aligned} \tag{20}$$

where × is a usual vector product in R^3, and the Lagrange multiplier is determined from the constraint $(N^+, \omega_+) + (N^-, \omega_-) = 0$. Equations (7) correspond to the case in which B_+ and B_- are diagonal, and $C = 0$.

If B_+, B_-, and C are diagonal, and $N^+ = \sigma_+ e_+^k$, $N^- = \sigma_- e_-^k$, $\sigma_\pm \in R, \sigma_+^2 + \sigma_-^2 > 0$ in the case of $\mathcal{G} \in \mathcal{A}$ (or $N^+ = \sigma_+ f_+^k$, $N^- = \sigma_- f_-^k$ in the case of $\mathcal{G} \in \mathcal{B}$), we can get from (6) that the EPS equations have an integral invariant. Then the non-holonomic constraint is

$$(N^+, \omega_+) + (N^-, \omega_-) = M_k^+(\sigma_+ B_+^k + \sigma_- C^k) + M_k^-(\sigma_+ C^k + \sigma_- B_-^k) = 0. \tag{21}$$

The measure could be preserved for other constraints as well, but under supplementary conditions for $B_\pm$ and C. We shall restrict ourselves to the case in which the constraint is given by (21).

Proposition 1. The Euler–Poincaré–Suslov equations on $\mathcal{G}^*$, where $\mathcal{G} \in \mathcal{A}$ or $\mathcal{G} \in \mathcal{B}$ with Hamiltonian (17) where B_+, B_-, C are diagonal, and the non-holonomic constraint (21) are integrable.

Proof. Without loss of generality, suppose $N^{\pm} = \sigma_{\pm} e^1_{\pm} (N^{\pm} = \sigma_{\pm} f^1_{\pm})$. First, let $\mathcal{G} \in \mathcal{A}$. Then

$$\mathrm{d}I_1(N) = 2qn_1\sigma_+ M_1^+ + 2n_1\sigma_- M_1^- \qquad \mathrm{d}I_2(N) = n_1\sigma_- M_1^+ + n_1\sigma_+ M_1^- . \tag{22}$$

If $\sigma_- = q = 0$, then by lemma 1, I_1 is the first integral of (19). Otherwise, from (21), (22), and lemma 1, it is

$$\begin{aligned} I = (2q\sigma_+^2 - 2\sigma_-^2)(\sigma_+ B_+^1 + \sigma_- C^1)(2\sigma_- I_2 - \sigma_+ I_1) \\ +(2\sigma_-^2 - 2q\sigma_+^2)(\sigma_+ C^1 + \sigma_- B_-^1)(2q\sigma_+ I_2 - \sigma_- I_1). \end{aligned} \tag{23}$$

Similarly, for $\mathcal{G} \in \mathcal{B}$, it can be proved that the invariant

$$I = m_1(\sigma_-\sigma_+ B_+^1 + \sigma_-^2 C^1) I_1 + n_1(\sigma_+\sigma_- B_-^1 + \sigma_+^2 C^1) I_2 \tag{24}$$

satisfies $\mathrm{d}I(N) = 0$ on the constraint (21), and it is the integral of equation (20). Thus, the task of integrating equations (19) and (20) is reduced to that of finding the third integral, independent of constrain (21), Hamiltonian (17), and the invariants (23) and (24). We are interested in the integral of the polynomial form

$$F_3 = x_+(M_2^+)^2 + y_+(M_3^+)^2 + 2fM_2^+M_2^- + 2gM_3^+M_3^- + x_-(M_2^-)^2 + y_-(M_3^-)^2. \tag{25}$$

By using the constraint we can express M_1^+ as a function of M_1^-. Then the equations for $M_2^{\pm}$ and $M_3^{\pm}$ take the form

$$\begin{aligned} \dot{M}_2^+ &= M_3^+ M_1^- L_1^+ + M_3^- M_1^- E_1^+ \\ \dot{M}_3^+ &= M_2^+ M_1^- L_2^+ + M_2^- M_1^- E_2^+ \\ \dot{M}_2^- &= M_3^- M_1^- L_1^- + M_3^+ M_1^- E_1^- \\ \dot{M}_3^- &= M_2^- M_1^- L_2^- + M_2^+ M_1^- E_2^- . \end{aligned} \tag{26}$$

Coefficients $L_1^{\pm}$, $L_2^{\pm}$, $E_1^{\pm}$, and $E_2^{\pm}$ are determined from (19) and (20). Now, we find that the condition $\dot{F}_3 = 0$ is equivalent to the following system of linear equations for $x_{\pm}$, $y_{\pm}$, f, and g:

$$\begin{aligned} x_+ L_1^+ + y_+ L_2^+ + fE_1^- + gE_2^- &= 0 \\ x_+ E_1^+ + y_- E_2^- + fL_1^- + gL_2^+ &= 0 \\ x_- L_1^- + y_- L_2^- + fE_1^+ + gE_2^+ &= 0 \\ x_- E_1^- + y_+ E_2^+ + fL_1^+ + gL_2^- &= 0. \end{aligned} \tag{27}$$

The system (27) always has a solution, and for general values of f and g the integral F_3 is independent of (21) and other integrals (17), (23), and (24). ☐

By the use of the physical meaning of the EPS equations, we find that special cases of our result are integrable perturbations of the classical Suslov problem (5).

Example 2. Equations (19) on $e(3)^*$ for $C = 0$, $N^- = 0$, $M^+ = M$, $M^- = \gamma$, $\omega_+ = \omega$, $B_+ = A$, and $B_- = B$ become

$$\dot{M} = M \times AM + \gamma \times \frac{\partial V}{\partial \gamma} + \lambda N \qquad \dot{\gamma} = \gamma \times \omega \qquad (N, AM) = 0$$

where $V = \frac{1}{2}(B\gamma, \gamma)$. This is the Suslov problem (5), with the additional axially-symmetric potential field $V(\gamma)$ (γ is a constant vector in a fixed reference frame). From proposition 1 we obtain the well known result: if N is an eigenvector of the operator A, then the Suslov problem with a quadratic potential is integrable [3, 4].

Example 3. Equations (20) on $so(4)^*$ describe the rotation of a rigid body with the elliptical hole, filled with ideal incompressible fluid. This problem, without constraint, was studied by Zhukovski, Poincaré and Steklov at the beginning of the century (see references in [12]). The total angular momentum and the vortex vector of the fluid are M^+ and M^-, respectively. For $N^- = 0$, the rigid body is subject to the same constraint as in the Suslov problem: $(N^+, \omega_+) = 0$, where N^+ is a constant vector, and ω_+ is the angular velocity of the rigid body. For the Hamiltonian we take

$$H = \tfrac{1}{2}(G^{-1}M^+, M^+) + (2d(CDG)^{-1}M^+, M^-) \\ +\tfrac{1}{2}((\tfrac{1}{5}mC^{-1} + 4d^2C^{-2}D^{-2}G^{-1})M^-, M^-)$$

where $I = \mathrm{diag}(I_1, I_2, I_3)$ is the inertia operator, $D = \mathrm{diag}(D_1, D_2, D_3)$ is the operator which maps the unit sphere to the ellopsoid, $d = \det D$, m is the mass of the fluid, $C = \mathrm{Tr}(D^2)E - D^2$, and E is the unit matrix, and $G = I + \frac{1}{5}mC^{-1}(C^2 - 4d^2D^{-2})$ [12].

4. Comments

Geodesic flows on $SO(4)$ are very well studied. Necessary conditions on metrics, for the integrability of equations (19) and (20), without non-holonomic constraints, could be found in [12–14] and [12, 15], respectively.

In the paper [9] there is a detailed analysis of the EPS equations for $SO(n)$ with $\frac{1}{2}(n-1)(n-2)$ constraints.

One of the main consequences of the symmetry of geodesic flows on G with left-invariant constraints is the reduction to the EPS equations on $\mathcal{G}^*$. This is part of the general reduction for the non-holonomic Lagrangian systems (Q, L, D) studied recently [17]. There it has been proved that if Q is a principal bundle, $Q \to Q/G$, with the Lagrangian and the distribution D invariant under the action of the group G, then the equations of motion could be considered on the reduced space D/G. Also, the equations written in the case of the 'principal' assumption ($\mathrm{span}\{D_x, T_x\mathrm{Orb}_G(x)\} = T_xQ$ for all $x \in Q$), where $Q = G$, the Lagrangian is just the kinetic energy, and $\ker(\alpha_g) \subset T_gG$ is the distribution D, coincide with the EPS equations.

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